THE OCTOBER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and eighty-sixth regular meeting of the Society was held in New York City on Saturday, October 28, 1916. The attendance at the morning and afternoon sessions included the following forty-two members:

Professor M. J. Babb, Dr. F. W. Beal, Mr. D. R. Belcher, Professor E. W. Brown, Dr. Emily Coddington, Professor F. N. Cole, Professor Elizabeth B. Cowley, Professor Louise D. Cummings, Dr. H. B. Curtis, Professor L. P. Eisenhart, Mr. G. W. Evans, Dr. C. A. Fischer, Professor T. S. Fiske, Dr. C. H. Forsyth, Professor O. E. Glenn, Dr. T. H. Gronwall, Professor C. C. Grove, Professor M. W. Haskell, Professor H. E. Hawkes, Professor Dunham Jackson, Dr. L. L. Jackson, Mr. S. A. Joffe, Professor Edward Kasner, Dr. J. R. Kline, Mr. E. H. Koch, Jr., Mr. Harry Langman, Professor H. B. Mitchell, Professor R. L. Moore, Mr. G. W. Mullins, Professor H. W. Reddick, Mr. J. F. Ritt, Mrs. J. R. Roe, Dr. Caroline Seely, Professor L. P. Siceloff, Professor D. E. Smith, Professor P. F. Smith, Professor Oswald Veblen, Dr. J. H. Weaver, Dr. Mary E. Wells, Professor H. S. White, Mr. J. K. Whittemore, Professor J. W. Young.

The President of the Society, Professor E. W. Brown, occupied the chair. The Council announced the election of the following persons to membership in the Society: Mr. A. C. Bose, Calcutta, India; Professor L. C. Emmons, Michigan Agricultural College; Professor A. M. Harding, University of Arkansas; Dr. W. L. Hart, Harvard University; Dr. J. R. Musselman, University of Illinois; Mr. S. Z. Rothschild, Sun Life Insurance Company, Baltimore, Md.; Professor Pauline Sperry, Smith College. Six applications for membership in the Society were received.

The Council submitted a list of nominations for officers and other members of the Council to be elected at the annual meeting. Committees were appointed to arrange for a joint session at the annual meeting with other scientific bodies and for the summer meeting of 1917.

The following papers were read at the October meeting:

(1) Mrs. J. R. ROE: "Interfunctional expressibility problems of symmetric functions."

(2) Professor E. D. ROE, JR.: "A geometric representation."

(3) Professor E. D. ROE, JR.: "Studies of the Kreisteilungsgleichung and related equations."

(4) Professor E. D. ROE, JR.: "The irreducible factors of $x^{n-1} + x^{n-2} + \cdots + 1$."

(5) Professor H. B. MITCHELL: "On the imaginary roots of a polynomial and the real roots of its derivative."

(6) Dr. J. H. WEAVER: "Some properties of parabolas generated by straight lines and circles."

(7) Professor F. N. COLE: "Complete census of the triad systems in fifteen letters."

(8) Professor O. E. GLENN: "Translation surfaces associated with line congruences."

(9) Professor O. E. GLENN: "Methods in the invariant theory of special groups, based on finite expansions of forms."

(10) Professor R. L. MOORE: "A theorem concerning continuous curves."

(11) Dr. J. R. KLINE: "The converse of the theorem concerning the division of a plane by an open curve."

(12) Mr. H. S. VANDIVER: "Note on the distribution of quadratic and higher power residues."

(13) Mr. H. S. VANDIVER: "The generalized Lagrange indeterminate congruence for a composite ideal modulus."

Professor Roe's first two papers were read by Mrs. Roe; his third paper, Professor Glenn's first paper, and Mr. Vandiver's papers were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. If α is a complex of numbers, b the elementary symmetric functions of the α's, S the power sums. and t the complete symmetric functions, then, of the complex α, Mrs. Roe undertakes to investigate the relations of the four kinds of functions, viz.: the monomial symmetric functions, the power sums, the elementary symmetric functions and the complete symmetric functions. If the four kinds of functions are taken two at a time, six combinations arise, and a consideration of the expressibility of each of a pair in terms of the other leads to a systematic study of twelve tables of coefficients.

The principal object of the investigation is to derive the properties of, and interfunctional relations and general formulas for, the coefficients in these tables.

The problems of the t's and the α's and also those of the S's and the α's are especially considered in this paper. Tables up to weight ten illustrate the results obtained.

2. Professor Roe adjoins the complex plane to the x axis of the real plane for the graphical realization of the complex values, when such exist, of y in $y = f(x)$, and also for the purpose of representing some real curves in space by a single equation in x and y. The complex plane is adjoined so that its origin is always in the axis of x with its axis of positive reals parallel to the positive direction of the axis of y, in fact in the real plane, as the complex plane slides along always perpendicular to the x axis at the distance x as x changes. By this representation every curve has an actual locus from $-\infty$ to $+\infty$. The method is applied to the function

$$y = \{x^{x-1}(x-1)^x\}^{1/(1-2x)}.$$

In the usual simplest real representation this consists of a curve in the real plane between $x = -\infty$ and $x = 0$, and $x = 1$ to $x = +\infty$, but between $x = 0$ and $x = 1$ consists only of discrete points. But if the complex value of y corresponding to a value of x is laid off on the complex plane adjoined to the x axis at the distance x, the curve between 0 and 1 becomes multiple valued for each simplest branch, continuous, and consists of an infinite number of spirals which pierce the real plane in the discrete points and all lie on a real surface of revolution $y^2 + z^2 = [\text{mod } \{x^{x-1}(x-1)^x\}^{1/(1-2x)}]^2$ where x, y, z are the coordinates of real space of three dimensions. Finally the general equation in x and y alone of all real spirals in space of three dimensions winding around the axis of x is shown to be

$$y = f_1(x)e^{f_2(x)i}.$$

3. In this paper Professor Roe studies the four equations $\cos n\varphi \pm \cos(n+1)\varphi = 0$, $\cos(n-1)\varphi \pm \cos(n+1)\varphi = 0$, obtaining the number of and expressions for their distinct roots and applying the results to some geometrical constructions. The "Kreisteilungsgleichung" in algebraic form (derivable from the second equation) is studied and an explicit expression is obtained for the general coefficient of any term in it. The same is also done for a like algebraic equation derivable from the first equation above.

4. In this paper Professor Roe considers the irreducible factors of $f(x, n) \equiv x^{n-1} + x^{n-2} + \cdots + 1$ by means of sums of powers of the roots of $(x - 1)f(x, n) = 0$.

5. The object of Professor Mitchell's paper is the deduction of simple criteria which may enable us to infer, without laborious calculation, the approximate position of a pair of imaginary roots of a polynomial with real coefficients, when the existence of imaginary roots is revealed through the vanishing of the derivative, (*a*) beyond the interval between the greatest and the least real root of the polynomial, (*b*) more than once in an interval between two consecutive real roots. Under the first assumption, upper and lower limits are determined which depend only upon the degree of the polynomial and the distance between a real root of the derivative and the adjacent real root of the polynomial. Under the second assumption it is shown that if the polynomial have not more than 8 imaginary roots, the real part of at least one pair must lie in the interval where the derivative vanishes more than once, and that the coefficient of i must be less in absolute value than the extent of that interval. When the polynomial has more than 8 imaginary roots, but when the number of real roots on one side of the interval is not greater than 8, it is shown that not all of the imaginary roots can have their real parts without and on one side of the interval —this restriction thus applying to all polynomials with real coefficients whose degree is less than 20.

The criteria are particularly suited to graphical analysis, as they involve only the approximate positions of the real roots and bend points.

6. Starting with the well-known fact that the locus of the center of a circle tangent to a given straight line and a given circle is two parabolas (one of which may be degenerate), Dr. Weaver proves several theorems showing relations between the three curves and involving questions of collinearity and concurrence. He then extends some of these results to the triangle and its circles and sets forth some sets of perspective triangles for which the axis of perspectivity passes through the center of perspectivity.

7. Professor H. S. White has determined all the triad sys-

tems in fifteen letters that have groups of substitutions into themselves, the number of types being 44.* Professors White and Cummings have also identified 33 groupless systems in fifteen letters, but a complete census of the groupless systems remained to be carried out. No method of doing this is known except that of actual sifting of all possible cases.

In forming triad systems in fifteen letters 1, 2, 3, $\cdots$, 15 the seven triads that contain 1 may be written in the style 1 2 3, 1 4 5, 1 6 7, 1 8 9, 1 10 11, 1 12 13, 1 14 15. For the triads containing 2 there are then four types of opening

(1) 2 4 6, 2 5 7, 2 8 10, 2 9 11, 2 12 14, 2 13 15;
(2) 2 4 6, 2 5 7, 2 8 10, 2 11 12, 2 13 14, 2 9 15;
(3) 2 4 6, 2 7 8, 2 5 9, 2 10 12, 2 13 14, 2 11 15;
(4) 2 4 6, 2 7 8, 2 9 10, 2 11 12, 2 13 14, 2 5 15;

which may be designated as the triple tetrad, single tetrad, hexad, and duodecad types, respectively. Professor Cole has found that the only triad system in fifteen letters that can be made up from hexads and duodecads exclusively is the known system of Heffter. All others involve tetrads.

If to the tetrad 1 2 3, 1 4 5, 1 6 7; 2 4 6, 2 5 7 there be added the triads 3 4 7, 3 5 6, the result is a triad system in seven letters within that of the fifteen letters, which is then said to have a 7-head, or merely head. If 3 5 6 is replaced by 3 5 8, we may perhaps speak of a semi-head. The complete census of the triad systems in fifteen letters may then be tabulated as follows:

I. Types with triple tetrads, 60, of which 22 have heads, 12 have semi-heads but no heads, and 26 have neither head nor semi-head.

II. Types with tetrads but no triple tetrads, 19, of which 1 has a head, 3 have semi-heads but no heads, and 15 have neither head nor semi-head.

III. Heffter's headless system, 1.

Total, 80 systems.

The 23 types with heads all have groups; of the headless types 21 have groups and 36 are groupless.

8. Professor Glenn's first paper appeared in full in the December BULLETIN.

9. A paper by Professor Glenn published in the *Trans-*

* See this BULLETIN, vol. 21, pp. 164, 488.

actions in 1914 contains the determination of the expansion of the general binary form f of order $n(m+1)-1$ as a binary m-ic in two arbitrary forms f_{1n}, f_{2n} of order n, the coefficients of the expansion being quantics φ_{in-1} $(i = 0, \cdots, m)$ of order $n-1$. Assuming, in the present paper, that f_{1n}, f_{2n} are the universal covariants of a definite linear group G, it is shown that the φ_{in-1} are covariants; and a complete system of f under G is the simultaneous system of the set φ_{in-1}, f_{1n}, f_{2n}. Particular cases are treated; the most elementary being an explicit derivation of the (known) system of orthogonal concomitants of an m-ic.

As a result of applying methods similar to the above in the case when G is the group of binary linear transformations modulo p, $p \neq 2$, a superior limit for the order of an irreducible formal covariant modulo p of any binary form or set of forms is determined.

10. Professor Moore proposes to show that every two points of a continuous curve (no matter how crinkly it may be) can be joined by a *simple* continuous arc that lies wholly in the curve.

11. In his paper* "On the foundations of plane analysis situs," Professor R. L. Moore defines an open curve as a closed connected set of points M such that if P is any point of M, then $M-P$ is the sum of two mutually exclusive connected sets of points, neither of which contains a limit point of the other. He shows that if l is an open curve and S is the set of all points, then $S-l=S_1+S_2$, where S_1 and S_2 are connected point sets, such that an arc from a point of S_1 to a point of S_2 contains at least one point of l. This theorem is proved on the basis of his set of axioms Σ_3 and is therefore true in certain spaces which are neither metrical, descriptive nor separable.

Dr. Kline proves the converse of this theorem for open curves. The statement of the converse theorem is as follows:

Suppose K is a closed set of points and that $S-K=S_1+S_2$ where S_1 and S_2 are non-compact point sets such that

(1) every two points of S_i $(i = 1, 2)$ can be joined by an arc lying entirely in S_i.

(2) every arc joining a point of S_1 to a point of S_2 contains a point of K.

* *Transactions*, vol. 17, no. 2, p. 131 (April, 1916).

(3) if O is a point of K and P is any point not belonging to K, then P can be joined to O by an arc having no point except O in common with K.

Every point set K that satisfies these conditions is an open curve.

12. If p is a prime of the form $4n+3$, consider the number of quadratic residues included in the set $1, 2, \cdots, 2n+1$. In the present note Mr. Vandiver proves a theorem which sets forth a connection between this number and the number of quadratic residues in any set defined by $h+[ah]<p$, where a is a fixed integer less than $p-1$ and h ranges over the set $1, 2, \cdots, p-1$, the expression $[ah]$ denoting the least positive residue of ah, modulo p. Analogous theorems are also found concerning the distribution of higher power residues.

13. Consider the indeterminate congruence of Lagrange

$$(x-1)(x-2)\cdots(x-(p-1)) \equiv x^{p-1}-1 \pmod{p},$$

where x is an indeterminate and p is a prime integer. Mr. Vandiver obtains some generalizations of this relation such that the set $1, 2, \cdots, p-1$ modulo p is replaced by all the incongruent residues of a composite ideal modulus which are prime to the modulus. The paper will appear in the *Annals of Mathematics*.

F. N. COLE,
Secretary.

CORRECTION.

THE following regretable errata in the reports of the summer meeting and colloquium of the Society, published in the November BULLETIN, have been brought to the attention of the Secretary:

I. In the report of the summer meeting, page 65, it is stated that Professor C. N. Moore's paper appeared in full in the October BULLETIN. A paper with the same title did appear in the October BULLETIN, but it was read at the annual meeting held last January. The abstract of Professor Moore's summer meeting paper is printed below, with apologies to the author.

10. In Professor Moore's paper an example is given of a function continuous in an interval $0 \leqq x \leqq 1$, whose development in Bessel's functions is not summable $(C\lambda)$ at the point $x = 0$, for any value of λ in the interval $0 \leqq \lambda < \frac{1}{2}$.

11. In the report of the colloquium, pages 85–88, Professor Veblen's subject matter appears distributed in six "Lectures," whereas only five lectures were actually delivered by each author. The headings in Professor Veblen's synopsis represent certain divisions of the subject matter, not the division into lectures, and the word "Lecture" should have read "Section."

THE MAXIMUM NUMBER OF CUSPS OF AN ALGEBRAIC PLANE CURVE, AND ENUMERATION OF SELF-DUAL CURVES.

BY PROFESSOR M. W. HASKELL.

(Read before the San Francisco Section of the American Mathematical Society, October 24, 1914.)

IT is well known that the double points of a rational algebraic curve can not in general all be cusps, and the maximum number of cusps has been determined in certain special cases. It is not difficult to find the maximum number from the consideration that none of the numbers in Plücker's equations can be negative.

Let m be the order and n the class, d the number of double points, k of cusps, i of inflexions and t of double tangents. We may first assume $d = 0$. In this case

$$2t = [m^2 - 9 - 3k][m^2 - 2m - 3k]$$

and the following inequalities must be satisfied:

(1) $$3k < m(m-1),$$

(2) $$8k \leqq 3m(m-2),$$

(3) $$2k \leqq (m-1)(m-2),$$

(4) Either $3k \leqq m^2 - 9$ and $3k \leqq m^2 - 2m$ simultaneously or else

(5) $3k \geqq m^2 - 9$ and $3k \geqq m^2 - 2m$ simultaneously.

Now, if m is greater than 9, $m^2 - 9$ is greater than $m(m-1)$ while if $m = 7, 8, 9$, $m^2 - 9$ is greater than $\frac{8}{3}m(m-2)$, so that if m is greater than 6, (4) must be satisfied. Hence, if m is greater than 6,

$$k \leqq \tfrac{1}{3}m(m-2).$$

Indeed, this formula is satisfied for $m = 3$ and $m = 5$, as is well known. The only exceptions are therefore $m = 4$ and $m = 6$.

Now, if m is of the form $3k$ or $3k + 2$, it is easy to show that a curve with the maximum number of cusps can have no further double points. The curve is self-dual, and

$$n = m, \quad d = t = 0, \quad k = i = \tfrac{1}{3}m(m-2),$$

$$p = \tfrac{1}{6}(m-2)(m-3).$$

If, however, m is of the form $3k + 1$, the curve may have one double point in addition. This curve is also self-dual, and

$$n = m, \quad d = t = 1, \quad k = i = \tfrac{1}{3}[m(m-2) - 2],$$

$$p = \tfrac{1}{6}(m-1)(m-4).$$

This result allows us to enumerate all self-dual curves. For, if $n = m$,

$$k = i = m - 2 + 2p,$$

$$d = t = \tfrac{1}{2}(m-2)(m-3) - 3p,$$

so that the minimum number of cusps for a self-dual curve is $m - 2$, and there is hence just one self-dual curve for every value of m corresponding to each value of p from zero up to the value given above.

Moreover, since

$$k = m - 2 + 2p \leqq \tfrac{1}{3}m(m-2),$$

the lowest order of a self-dual curve of given deficiency $p(>0)$ is the smallest integer satisfying the inequality

$$m \geqq \frac{5 + \sqrt{24p+1}}{2}.$$

NEW YORK,
November, 1916.

NOTE ON ASYMPTOTIC EXPRESSIONS IN THE THEORY OF LINEAR DIFFERENTIAL EQUATIONS.*

BY PROFESSOR W. E. MILNE.

(Read before the American Mathematical Society, December 28, 1915.)

LET n independent solutions of the linear differential equation

$$(1) \qquad \frac{d^n u}{dx^n} + P_2(x)\frac{d^{n-2}u}{dx^{n-2}} + \cdots + P_n(x)u + \rho^n u = 0$$

be denoted by $y_1, y_2, \cdots, y_n$. It is the aim of this note to establish asymptotic representations of a particular form for the n functions $\bar{y}_1, \bar{y}_2, \cdots, \bar{y}_n$ determined by the n identities

$$(2) \qquad \sum_{i=1}^{n} y_i^{(s-1)}\bar{y}_i = \begin{cases} 0, \text{ if } s = 1, 2, \cdots, n-1, \\ 1, \text{ if } s = n. \end{cases}$$

For this purpose we employ asymptotic forms for the y's, as follows.† If the coefficients $P_s(x)$ in (1) have continuous derivatives of order $(m + n - s)$, m being a positive integer or zero, in the interval $a \leqq x \leqq b$, then there exist n independent solutions of (1) of the form

$$(3) \qquad \begin{aligned} y_i &= u_i(x, \rho) + e^{\rho w_i(x-c)}E_{i0}/\rho^{m+1}, \\ y_i^{(k)} &= u_i^{(k)}(x, \rho) + e^{\rho w_i(x-c)}E_{ik}/\rho^{m+1-k} \\ &\qquad (i = 1, 2, \cdots, n; k = 1, 2, \cdots, n-1), \end{aligned}$$

where

$$(4) \qquad u_i(x, \rho) = e^{\rho w_i(x-c)}\left[1 + \frac{\varphi_1(x)}{\rho w_i} + \cdots + \frac{\varphi_m(x)}{(\rho w_i)^m}\right].$$

The functions $\varphi_j(x)$ have continuous $(m + n - j)$th derivatives in (a, b) and are independent of i, while for x in (a, b)

* The formulas given here were published without proof in the *Proceedings Nat. Academy of Sciences*, vol. 2 (1916), pp. 543–5.

† The existence of asymptotic solutions of (1) in nearly the form given in (3) was proved by Birkhoff, *Transactions Amer. Math. Society*, vol. 9 (1908), pp. 219–231, 381–2. The proof of the formulas in (3) and (4) is conducted in a similar manner and offers no essentially new difficulty. For an explicit statement of the difference between Birkhoff's formulas and those here given, see the note in the *Proceedings* referred to above.

and ρ in a suitably chosen sector* S_k of the ρ-plane the E_{ij} are analytic in ρ and bounded as ρ becomes infinite.

By the use of the above formulas we can now show that *when the coefficients $P_s(x)$ have continuous $(m+n-s)$th derivatives in (a, b), the n functions $\bar{y}_i$ have when $|\rho|$ is large the asymptotic form*†

$$\bar{y}_i = \frac{e^{-\rho w_i(x-c)}}{n(\rho w_i)^{n-1}}[\bar{v}_i(x, \rho) + \bar{E}_i/\rho^{m+1}] \qquad (i = 1, 2, \cdots, n), \tag{5}$$

where

$$\bar{v}_i(x, \rho) = 1 + \psi_1(x)/\rho w_i + \cdots + \psi_m(x)/(\rho w_i)^m. \tag{6}$$

The functions $\psi_j(x)$ are independent of i and have continuous $(m+n-j)$th derivatives in (a, b), and for x in (a, b) and ρ on S_k the $\bar{E}_i$ are analytic in ρ and bounded as ρ becomes infinite.

The method of proof is to substitute in (2) from (5) and (6) and show that the m ψ's and n $\bar{E}$'s can be chosen to satisfy (2). It will readily be seen that $y_i^{(k)}$ can be written

$$y_i^{(k)} = (\rho w_i)^k e^{\rho w_i(x-c)}[v_{ik}(x, \rho) + E_{ik}'/\rho^{m+1}], \qquad (i = 1, 2, \cdots, n;\ k = 0, 1, \cdots, n-1), \tag{7}$$

where

$$v_{ik}(x, \rho) = 1 + \varphi_{1k}/\rho w_i + \cdots + \varphi_{mk}/(\rho w_i)^m, \tag{8}$$

and

$$\varphi_{jk} = \sum_{r+t=j}\left[\frac{k!}{r!(k-r)!}\right]\varphi_t^{(r)}.$$

Substitute in (2) from (5) and (7), cancel the exponentials, and divide out the factor $-1/n\rho^{n-s}$ from the sth equation. The result is ‡

$$\sum_{i=1}^{n} w_i^s v_{i,s-1}\bar{v}_i + \frac{1}{\rho^{m+1}}\left[\sum_{i=1}^{n}(w_i^s + E/\rho)\bar{E}_i + E\right] = \begin{cases} 0, \text{ if } s = 1, 2, \cdots, n-1, \\ -n, \text{ if } s = n. \end{cases} \tag{9}$$

* For the precise definition of S_k see either Birkhoff, loc. cit., or the note in the *Proceedings*.

† A similar but less explicit formula was used by Birkhoff, loc. cit., p. 391, formula (56).

‡ The letter E is here used in a generic sense to denote any function analytic in S_k and bounded in S_k as ρ becomes infinite.

By (6) and (8)

$$\sum_{i=1}^{n} w_i^s v_{i,s-1}\bar{v}_i = \sum_{i=1}^{n}\sum_{j=0}^{m} \theta_{js}\rho^{-j} w_i^{s-j} + E/\rho^{m+1}, \tag{10}$$

where

$$\theta_{js} = \sum_{r+t=j} \varphi_{r,s-1}\psi_t.$$

Now $\sum_{i=1}^{n} w_i^{s-j}$ is zero except when $j - s = kn$, where k is integral, in which case it is $(-1)^k n$. Therefore, if we denote $kn + s$ by z, the double sum in (10) reduces to

$$\sum_{k=0}^{I_s} (-1)^k n\theta_{zs}/\rho^z$$

when $s = 1, 2, \cdots, n-1$, and to

$$-n + \sum_{k=0}^{I_s} (-1)^k n\theta_{zs}/\rho^z$$

when $s = n$. The symbol I_s denotes the integral part of $(m - s)/n$. Substituting back into (9), we now get

$$\sum_{k=0}^{I_s} (-1)^k n\theta_{zs}/\rho^z + \frac{1}{\rho^{m+1}}\left[\sum_{i=1}^{n} (w_i^s + E/\rho)\bar{E}_i + E\right] = 0$$

$$(s = 1, 2, \cdots, n).$$

Since these are identities in ρ, the coefficient of every power of $1/\rho$ up to and including the mth must vanish separately, giving the m equations

$$\theta_{zs} = 0, \quad \text{or} \quad \psi_z = -\sum_{r+t=z}^{t<z} \varphi_{r,s-1}\psi_t \tag{11}$$

$$(k = 0, 1, \cdots, I_s;\ s = 1, 2, \cdots, n),$$

and leaving the n equations

$$\sum_{i=1}^{n} (w_i^s + E/\rho)\bar{E}_i + E = 0 \quad (s = 1, 2, \cdots, n). \tag{12}$$

From equations (11) the m functions ψ_j may be determined in succession, and are readily shown by mathematical induction to have continuous derivatives of order $(m + n - j)$ in (a, b). The functions represented by the letter E in equations (12) can be expressed in terms of the φ's, the ψ's, and the E_{ik}'s,

and are therefore now to be regarded as known functions. When $|\rho|$ is sufficiently large the determinant of the coefficients in (12) is not zero, so the $\bar{E}_i'$s can be uniquely determined. It is plain that for ρ in S_k they are analytic in ρ and bounded as ρ becomes infinite.

BOWDOIN COLLEGE,
October, 1916.

ON NOTATIONAL EQUIVALENCE.

BY PROFESSOR EDWIN BIDWELL WILSON.

IN reply to my query* to Dr. Poor "Why not make the work short?" he states† that brevity was not his aim, that one of his purposes was to exhibit the Burali-Forti and Marcolongo notation. I must accept that answer and admit my error in assuming that his only aim was to derive as directly as possible some transformations which are needed in certain studies in applied mathematics. It is, however, difficult for me to admit many of his other contentions. I have no desire to enter upon any polemic in regard to these matters, but it does seem that further explanation from Dr. Poor would be valuable to all who are interested in vectorial methods.

1. He states: That the use of words, such as grad, div, rot, is hampering seems to be a matter of opinion, since they may be used interchangeably with other symbols.

I hold that because two sets of symbols may be used as interchangeably as these and ∇ is no criterion at all that one is not more hampering than another. For instance, 94 and XCIV are equivalent symbols, so are 8 and VIII, and also 752 and DCCLII. Yet for the arithmetical operation of multiplying eight and ninety-four the Arabic notation is far superior to the Roman (or Greek); indeed so marked is the superiority that one may well wonder how far mathematics would now be advanced had no better system than the Roman been devised.

May we not fairly maintain that notationally Arabic and Roman numerals are not interchangeable? Is it true that two notations in terms of which premises and conclusions may both be stated are for that reason interchangeable? To

* Wilson, this BULLETIN, vol. 22, April, 1916, p. 336.
† Poor, this BULLETIN, vol. 22, July, 1916, p. 503.

me the whole series of formal operations by means of which the passage is made from premise to conclusion must be given a fundamental place in determining the question of notational interchangeability in so far as concerns the discussion of what is hampering and what felicitous.

What was Maschke's object in developing a symbolic method, analogous to that of Clebsch-Aronhold, for dealing with differential forms? It could hardly have been merely to introduce a new notation interchangeable with older ones with all the attendant liability to irritation and confusion.

2. Dr. Poor states that Burali-Forti and Marcolongo have pointed out how the dyadics of Gi' bs constantly depend on cartesian coordinates, a non-linear system.

Dr. Alexander Macfarlane called my attention to this remark (and others) in one of his letters written shortly before his death. He did not, nor do I, believe all the things those eminent authors say about Gibbs' system—or else I do not understand them. If it is meant that in the Gibbs-Wilson Vector Analysis many properties of dyadics are proved by means of cartesian coordinates, I should admit that some proofs of that sort were given, but should call attention to a subsequent work* also done after Gibbs' lectures, in which no such proofs are given. I have always taught my students, as I learned from Gibbs, that dyadics are in no way dependent on coordinates, no matter how proofs may be given. I am sorry if I have been propagating heterodoxies.

3. He states: It is unfortunate that Professor Wilson introduced cartesian coordinates into his proof since a coordinate system has no place in vector analysis.

To be sure, puristic ideals have enraptured the gaze of some students of vector analysis, as of some students of projective geometry and other branches of mathematics. The refinements of method which have sometimes resulted from this striving toward the puristic goal have been of great value. Whether, however, the insistence on such ideals under all circumstances is fortunate is a mooted question. Certainly it is possible to define cartesian coordinates in terms of vectors (a method recommended by Heaviside); it is then the coordinates which depend upon vector analysis and consequently have as much place in it as anything else.

* Wilson, *Trans. Conn. Acad. Arts Sci.*, New Haven, vol. 14, 1908, pp. 1–57.

The case in which I used cartesian coordinates in my note was in proving the operational identity $\int dS() = -\int d\tau()$. The proof of this well-known formula was not essential to my work—I could have merely quoted and applied the result as Dr. Poor quoted and applied many results from Burali-Forti and Marcolongo.*

4. He states: The notation of Burali-Forti and Marcolongo could have been made to compare very favorably with Professor Wilson's compact reproduction of the formulas in the Gibbs notation.

It would please me much to see this done. It is suggested that we need only compare the analytic statement of the theorems in the two notations. I am unable to make the comparison satisfactorily, or to admit that the comparison if made would be a just criterion. To my mind it would be necessary to compare the whole proofs in detail, and in particular to determine what propositions were needed for the proofs in the two notations which were not obvious consequences of the simple laws of operation in the respective notations.

For instance, Dr. Poor writes the relation†

$$\operatorname{div}_M \alpha \boldsymbol{u} = -\boldsymbol{u} \times \operatorname{grad}_P K\alpha,$$

$\boldsymbol{u}$ independent of M, which in Gibbs' notation is

$$\nabla_M \cdot (\alpha \cdot \boldsymbol{u}) = (\nabla_M \cdot \alpha) \cdot \boldsymbol{u} = -\boldsymbol{u} \cdot (\nabla_P \cdot \alpha),$$

and is obvious (it being understood that $\nabla_M = -\nabla_P$). But the relation is derived by Dr. Poor from a quoted formula

$$\operatorname{div}_M \alpha \boldsymbol{u} = \boldsymbol{u} \times \operatorname{grad}_M K\alpha + I_1\left(\alpha \frac{d\boldsymbol{u}}{dM}\right)$$

from Burali-Forti and Marcolongo. It may be that there is some way of remembering all such formulas in their notation, but I have never deciphered any. In the Gibbs notation we should have

$$\nabla \cdot (\alpha \cdot \boldsymbol{u}) = (\nabla \cdot \alpha) \cdot \boldsymbol{u} + \nabla_1 \cdot (\alpha \cdot \boldsymbol{u}_1) = (\nabla \cdot \alpha) \cdot \boldsymbol{u} + \nabla \boldsymbol{u} : \alpha$$

by the usual method of differentiation in situ.

* There seem to be ten such citations. See, Poor, this BULLETIN, vol. 22, January, 1916, pp. 174–181.

† Loc. cit. above, p. 178.

The important thing here is fundamental to the whole question of notation and particularly to notational interchangeability. The rule of differentiation in situ and the ordinary rules for the use of dot and cross in vector algebra taken with the identity $\int dS() = -\int d\tau()$ suffice to prove all Dr. Poor's theorems and many others of the sort without reference to any list of formulas—the whole thing has become mere formal operation which for a student of Hamilton, Tait, Gibbs, and McAulay is in the same category as the work

$$a - \frac{1}{a} = \frac{a^2 - 1}{a} = \frac{(a+1)(a-1)}{a}$$

is for the schoolboy.* If this is equally true of the student of Burali-Forti and Marcolongo, I am both surprised and happy.

ON PIERPONT'S INTEGRAL. REPLY TO PROFESSOR PIERPONT.

BY PROFESSOR MAURICE FRÉCHET.

My single aim in my previous contribution to this journal ("On Pierpont's definition of integrals," volume 22, number 6, March, 1916) was to point out that, in my own words, *this new definition is inappropriate. I still hold to my original assertion* (though for partly different reasons) and will show why I do so.

Thus the question whether two non-measurable sets with no points in common are separated or not is far from being the vital point. This being explicitly stated, I hasten to say that *concerning this last particular question,* Professor Pierpont is entirely justified in saying: "Professor Fréchet has been misled at this point . . . and his example establishes not an error on my part but a carelessness of reasoning on his." As a matter of fact, I too quickly assimilated in my mind "separated" with "having no point in common." The same thing occurred with the word "exterior" and my objection to theorem 341, page 346 arose from a miscon-

* It would not have been obvious to the schoolboy, perhaps not even to a professional mathematician, in the days before a suitable notation for elementary algebra had been developed.

ception of the meaning of this word in Pierpont's terminology. Indeed, as I was at the front (where I am still) when I wrote this article, I could only compose it out of notes formerly taken. And even now, I do not understand what Professor Pierpont describes as "sets exterior to each other." I will then let fall my objections to formulas (2), (5), (7), in Professor Pierpont's reply.

However, under present circumstances (I write this on June 30) it is wiser not to postpone my answer. And with but the information I have in hand, I will make good my point as follows:

The main differences between Lebesgue and Pierpont integrals are two in number.

I. When E is measurable and $f(x)$ is summable the common value of Lebesgue and Pierpont integrals $\int_E f(x)dx$ is arrived at in different manners.

Then, I still maintain that in this case the real difference between their definitions is *not* that—as Professor Pierpont asserts—he makes use of an infinite instead of a finite number of parts δ_i of E (as in Riemann's definition). It lies essentially in the use of measurable sets instead of intervals. For, in the most important case: when $f(x)$ is bounded, the finiteness or infiniteness of the number of parts δ_i of E is indifferent in Pierpont's definition. This is easily seen, starting from Pierpont's theorem, that the remainder

$$\overline{\text{meas.}}\ \delta_i + \overline{\text{meas.}}\ \delta_{i+1} + \cdots$$

(of a series equal to $\overline{\text{meas.}}\ E$) converges to zero.

It is open to Professor Pierpont to prove that this former assertion of mine is wrong.

I mention in passing that apropos of a different memoir, M. Lebesgue kindly pointed out to me that a definition of Lebesgue's integrals by means of Riemann's sums was given as early as 1905 by W. H. Young (*Philosophical Transactions*).

II. Pierpont's definition enables him to give a definite value to integrals which are not integrable according to M. Lebesgue.

Whereas when E is non-measurable, M. Lebesgue contents himself with saying that the measure of E is contained between $\overline{\text{meas.}}\ E$ and $\underline{\text{meas.}}\ E$, Professor Pierpont goes further. According to his definition, the function $f(x) \equiv 1$ is integrable

on any set E and

$$\int_E dx = \overline{\text{meas. } E}.$$

It is questionable *whether this precise formula is a decided improvement over M. Lebesgue's statement.* But, further, from this formula it is deduced that *the Pierpont integral does not enjoy the fundamental property that if E, F are sets with no points in common*

$$\int_{E+F} f(x)dx = \int_E f(x)dx + \int_F f(x)dx$$

(which however is true when E, F are "separated," according to Professor Pierpont). It suffices to apply this formula when $f(x) \equiv 1$, $E + F$ is an interval and E is non-measurable.

A REPLY TO A REPLY.

BY PROFESSOR JAMES PIERPONT.

As I view the issue between Professor Fréchet and myself, it may be summed up as follows:

1°. Professor Fréchet thought that it was possible to split a measurable set into two *separated* non-measurable sets, and he gave an alleged example. Since no such division is possible this example proved to be an ignis fatuus.

2°. Supported by this example, it was easy for Professor Fréchet to bring a number of grave charges against my work, in fact it might seem as if my whole theory had toppled to the ground.

3°. Professor Fréchet now admits (provisionally) that he was in error on this score, but he still holds to his "original assertion" that my integral definition "is inappropriate," "though for partly different reasons." What are these new reasons? Although I have read and reread the above article I have found but one, viz.: Suppose A is *non*-measurable and suppose B and C form a *non*-separated division of A, then the relation

$$\int_A = \int_B + \int_C \tag{1}$$

may not hold.

This very obvious fact I have known from the start; it is one that any one would discover. To Professor Fréchet this may be an insuperable objection and I have no contention with any one who holds this view. A similar peculiarity is presented in many theories. For example, one may take the stand that the double series

$$\begin{aligned} & a_{11} + a_{12} + a_{13} + \cdots \\ & + a_{21} + a_{22} + \cdots \\ & + a_{31} + \cdots \\ & \cdots \end{aligned} \tag{2}$$

is convergent only when it is absolutely convergent,* but many authorities do not. In the latter case one is led to a number of unexpected results; e. g., the series (2) may converge although every series formed of a row or a column of (2) is divergent.

To my mind it does not seem wise to be doctrinaire in such matters. The relation (1) does hold for separated divisions of A, and when A is measurable no other divisions are possible. Since no one as yet has exhibited a non-measurable set, only the existence of such sets having been established, it seems at least premature to argue on a priori grounds against any theory which makes a step in advance.

In any case the nature of Professor Fréchet's objections has been widely changed; as first formulated they struck at the very foundation of my theory by impeaching the correctness of one of its main theorems; at present the only objection I see is an expression of a personal opinion.

4°. Polemics are apt to be interminable; fresh charges are made, fresh rejoinders necessitated and so on ad infinitum. I therefore am not astonished that Professor Fréchet has injected a new element into the discussion. It now seems that the vital point is the "real difference" between Lebesgue's definition and my own. To me this question is one of complete indifference and I leave Professor Fréchet to settle it entirely to his own satisfaction.

* Cf. C. Jordan, Cours d'Analyse, vol. I, p. 302.

SHORTER NOTICES.

The Teaching of Algebra (including Trigonometry). By T. PERCY NUNN, M.A., D.Sc. London, Longmans, Green and Company, 1914. xvi + 616 pp. Price $2.

Exercises in Algebra (including Trigonometry). By T. PERCY NUNN. Part I. 1913. xii+422 pp. Price $1.10. Part II. 1914. xii+552 pp. Price $1.75.

A FEW years ago one would have been justified in expressing surprise had an American teacher of mathematics gone to England to study modern tendencies in secondary education. To-day, however, if one were to study the present efforts at reform in the teaching of mathematics, he would certainly go to England for some of the most prominent leaders in the movement. Fifteen years have wrought a great change, and while the change has not been along the exact lines suggested by Professor Perry there is no doubt that he was the most influential of those who awakened the teachers of mathematics in the English-speaking world from the lethargy into which they had fallen. Naturally, the movement to better the work in mathematics has not been uniformly successful; for many ill-considered attempts have been made in addition to the few which give reasonable promise of producing satisfactory results. England has had her extremists as America has had hers, but on the whole she has been more consistent than we in her insistence upon a high degree of scholarship.

Among the strongest advocates both for reform in teaching and for a maintenance of high scholarship is Dr. Nunn, vice-principal of the London Day Training College, a teacher of experience and a man who adds character to the title of educator. In the publication of the three works under review, Dr. Nunn has doubtless been assisted by the judgment of the editors of the series of which these books are a part, Messrs. Abbott and Jackson and Dr. Macaulay, three of the best-known leaders in the progress of secondary mathematics in England.

The general introduction considers the nature of algebra, the formula, the graph, and the questions of method and the curriculum. Part I first discusses non-directed numbers, then directed numbers, and finally logarithms. Part II is concerned chiefly with the trigonometry of the sphere (in

particular, projections), complex numbers, periodic functions, limits, and statistics. Each of these topics is fully amplified by the accompanying exercise manuals where abundant material is given for the student's mastery of the respective chapters.

To the American teacher it will be of chief interest, so far as a review is concerned, to have attention called to a few of those features of the books which reveal the new and richer field which it is proposed to open to the English student. One of the advantages of so doing is that we Americans may see a serious effort at improving algebra without making the subject so soft and insipid as to have no interest for any healthy student. If the reader shall conclude that the teaching of algebra, as here treated, means merely the teaching of Dr. Nunn's algebra, this will not militate against the fact that the book will be found suggestive and helpful to teachers who follow courses which are quite at variance with the one which the author here lays down.

In the first place the reader will be struck by the fact that Dr. Nunn is not merely a mathematician but a philosopher and a psychologist as well. A single excerpt will reveal this fact, but it may be added that the fact is commonly apparent as one reads the pages of the work. "Mathematical truths always have two sides or aspects. With the one they face and have contact with the world of outer realities lying in time and space. With the other they face and have relations with one another. . . . From its dim beginnings by the Euphrates and the Nile mathematics has been on the one hand a means by which man has constantly increased his understanding of his environment and his power of manipulating it, and on the other hand a body of pure ideas, slowly growing and consolidating into a noble rational structure. . . . Our purpose in teaching mathematics in school should be to enable the pupil to realize, at least in an elementary way, this twofold significance of mathematical progress. A person, to be really 'educated,' should have been taught the importance of mathematics as an instrument of material conquests and of social organization, and should be able to appreciate the value and significance of an ordered system of mathematical ideas. There is no need to add that mathematical instruction should also aim at 'disciplining his mind' or giving him 'mental training.' So far as the ideals intended by these phrases are

sound they are comprehended in the wider purpose already stated. . . . The theoretical questions which are of most importance in an elementary course are just those which arise naturally out of attempts to apply mathematical ideas and methods to practical purposes."

With this principle to guide him, Dr. Nunn proposes to consider those parts of algebra which have the highest practical value, and if at times he seems to depart from the path he lays out it should be said that this is with the definite intent to develop his theoretical discussions from a consideration of the practical. In America we have the pedantic term "motivation" to designate this method of approach to the theoretical, but it adds nothing to the idea.

The author's idea is the sound one that the point of approach to algebra is the formula, "illustrations being drawn largely from the 'Pocket Book' of the engineer and similar formularies. . . . A little later comes the study of those manipulations of a formula by which it may be made to yield truths unknown or unperceived before. . . . The gradual elaboration of the formula as an instrument of description and investigation is, then, the first business of the course in algebra."

To carry out this idea the author proposes a fusion plan which has distinct advantages, differing herein from the ill-considered ideas occasionally met of fusing such subjects as algebra and geometry. His idea is that trigonometry, being largely a science of formulas and identities, should grow out of algebra, and that the calculus should do the same. These ideas seem practical, although the details of their execution may vary under such conditions as our colleges impose upon the secondary schools. Carrying out his plan, the author not only introduces work in plane trigonometry, but proceeds to the trigonometry of the sphere, Mercator and great-circle sailing, and map projections. His study of complex numbers develops into a study of circular functions, this into periodic functions, this into progressive and stationary wave motion, and this into hyperbolic functions. The course ends with a very satisfactory presentation of limits, including the elements of differentiation and integration, and with a helpful introduction to the mathematics of statistics.

In the work in graphs Dr. Nunn takes a very sane stand. The extremes of a few years ago find no place in his plan. He recognizes the value of the graph, and gives it a large place

in his scheme, with many examples to show its use; but he also recognizes that "it is inferior to the symbolic formula in many important respects. Its accuracy depends largely upon mechanical or non-intellectual conditions, such as the skill of the draughtsman and the exactness of the squared paper. It is less compact and less easily reproduced. Its message is frequently inarticulate and obscure. For these and similar reasons it should be regarded as a subsidiary algebraic instrument which fulfills its best office when it either leads up to a formula by which it may itself be superseded, or serves to unfold more fully the implications of a formula whose properties have been only partially explored."

After such a carefully considered statement the reader will probably be surprised that Dr. Nunn should resort to graphic treatment in certain cases where it is, to say the least, of doubtful value. Such cases are his elaborate development of the identities $ac + bc = (a + b)c$, $ac - bc = (a - b)c$, and especially $a^2 - b^2 = (a + b)(a - b)$, and the colored graphs in connection with negative numbers.

Another feature in Dr. Nunn's scheme that will strike the reader as of very doubtful value is his postponement of the negative number. Under the older plan of teaching, this delay would have been desirable; but the facility with which the negative number is now illustrated, and the ease with which pupils grasp the idea, renders such delay unnecessary. If the theory is to be presented as scientifically as the author proposes, there is some reason for waiting, but it is probable that the loss exceeds the gain at this stage of algebra.

It is not desirable in a review of this nature to consider further the large number of details that are found in three volumes aggregating over 1,500 pages. The present review has already extended beyond the limits usually allowed in the BULLETIN, but it is proper to add that there has never before appeared a work so helpful to a teacher of algebra. The reviewer does not believe in the sequence that Dr. Nunn lays down; he does not believe in certain uses of the graph that are set forth; he does not believe in several of the efforts at "motivation" which are advanced; and he does not believe that we are ready, in this country at any rate, to undertake certain parts of the course which is planned. But on the other hand he has not failed to recommend the work to hundreds of teachers of mathematics as the most noteworthy

contribution to the teaching of algebra that has yet appeared. Teachers who know our American problem, who are well enough balanced not to be enticed into fields that are certain to resist cultivation at the present time in this country, and who are searching for sane methods of reform, should read the pages which Dr. Nunn has here written with such care and erudition, and with such force and clearness.

DAVID EUGENE SMITH.

Syllabus of Mathematics. A symposium compiled by the Committee on the Teaching of Mathematics to Students of Engineering. Published by the Society for the Promotion of Engineering Education, Ithaca, N. Y., 1912. 136 pp.

THE purpose of this syllabus is to collect those principles and methods of mathematics which should constitute the minimum mathematical equipment of the student of engineering, or "those things for which a student ought never to be obliged to refer to any book—the things which he should have constantly at his fingers' ends." The book contains separate syllabi on elementary algebra (14 pages), elementary geometry and mensuration (7 pages), plane trigonometry (19 pages), analytic geometry (28 pages), differential and integral calculus (44 pages), and complex quantities (3 pages) together with a report of the discussion of these syllabi at the Pittsburgh meeting of the Society (14 pages).

Such syllabi will be of great value to students or teachers who wish to review the essentials of elementary mathematics courses. They will especially aid those students who end courses without proper perspectives. However, some may see danger of low standards in so much stress upon the *minimum* mathematical equipment of an engineer. The ideal of the committee is hardly realized, for few engineers "know by heart" and never need to "look up in a book" all the material in these syllabi. While admitting the value of such synopses of minimum essentials, yet as pointed out in one discussion (page 126), there is also need for lists of *all* the topics and principles that should be included in mathematics courses for engineers since there is some danger of too little rather than too great mathematical equipment.

Only few illustrative problems appear. A valuable supplement would be two sets of problems; one set giving an illustrative problem corresponding to each principle; the

other set giving problems without specifying the particular principles involved. The committee is firmly of the opinion that the principal part of the engineering mathematics course should be problems worked by the students.

When the important principles of elementary algebra may be summed up in fourteen pages, it is surprising to find such universal weakness in algebra among our college students. In addition to the usual topics of algebra are found logarithms, inequalities, and the progressions. The part on exponents would be clearer to students if accompanied by translated word forms, for the full meaning of symbolic statements is often missed. Under neither algebra nor trigonometry is the distinction between equations and identities given. The operations which introduce or take out roots of an equation are not sufficiently brought out. An ingenious form for solving simultaneous linear equations is given.

Under geometry and mensuration we find properties of right and oblique triangles, angles in a circle, proportion of similar figures, area, volume and surface formulas, Cavalieri's theorem, theorems of Guldin or Pappus, and the prismoidal formula all given in seven pages.

The usual topics in trigonometry are given with commendable emphasis placed upon the sine, cosine, and tangent, thus making the other functions secondary. Line functions and graphic methods of remembering certain relations are prominent. The form of generalizing the definitions of the functions for any angle is to be commended.

Most of the part on analytic geometry is devoted to the conic sections. There is not sufficient emphasis placed upon methods of finding the equation when the locus is defined. Under neither algebra nor analytic geometry do we find sufficient treatment given to plotting and locating roots of numerical equations of higher degree; such theory as that usually given under the topic "theory of equations" in college algebra seems entirely missing. The part given to surfaces of the second degree is too condensed to be worth much. The part on the general equation of the second degree might well be condensed or even omitted in a syllabus giving only that which one "should have constantly at his fingers' ends."

The first section on the calculus is a good summary of functions and their graphic representation. In the definition of

function it would be better to emphasize the notion of *correspondence* rather than dependence. Graphs of the elementary functions are given and a short treatment of fitting functions to empirical data is included. In many places the notion of a limit is used but nowhere is limit defined. The fundamental relations of increments, derivatives and differentials are well presented. We are glad to find theorems on infinitesimals which are needed in setting up problems and which are frequently neglected when the calculus is approached from the standpoint of limits.

The part on differential equations seems too short. Integration by partial fractions is omitted. An illustrative problem would have emphasized the importance of the constant of integration. Some illustrations of the use of the definite integral are needed to make plausible their statement that "The concept of the definite integral is the most useful concept in the application of the calculus."

The promised parts on "Functions of Two or More Variables," "Numerical Computation," and on "Elementary Dynamics" have not yet appeared so far as the reviewer is aware.

ERNEST B. LYTLE.

Einführung in die höhere Mathematik. By HANS VON MANGOLDT. Vol. I: *Anfangsgründe der Infinitesimalrechnung und der analytischen Geometrie.* 1911. xiv+477 pp. Vol. II: *Differentialrechnung.* 1912. xi+566 pp. Vol. III: *Integralrechnung.* 1914. x+485 pp. Leipzig, S. Hirzel.

THE volumes before us constitute an important contribution toward the solution of a problem which is of great concern to the majority of teachers of mathematics. That problem has to do with the amount and arrangement of the mathematical methods to be included in courses for students of physics and of engineering. We are of course agreed that there must be included the analytical geometry of two and of three dimensions, differential and integral calculus. But though we realize the deficiencies of the traditional college courses in these subjects, we are not well agreed as to the remedy. There are on the one hand those who claim that the student in question need only learn when and how certain formulas should be used, on the other hand those who would teach all mathematical courses merely as if they were an

end in themselves, with no regard at all to the future interests of the student. Between these extremes the advocates of the middle course have had a hard time. Many more or less successful attempts have been made to solve the problem by combinations of the old courses, eliminating some of the established chapters here and there, and introducing the fundamental notions of analytic geometry and the calculus at earlier stages, mainly with a view to saving time that had hitherto been employed either in solving problems in several different ways, or in the study of subjects which had no immediate application to physics and engineering. Most of the combinations which have come to the reviewer's attention seem to him to fall short in the lack of a central idea which should weld the whole course into an harmonious whole. The old courses have been mixed without being unified.

The book under review is centered on the differential and integral calculus. The first volume is introductory. Assuming the student acquainted with elementary geometry, trigonometry and algebra, the author first discusses certain chapters in advanced algebra, viz. permutations and combinations, the binomial theorem, probabilities, determinants, irrational numbers, this latter including a clear and sound exposition of Dedekind's theory. These matters occupy 177 pages. He then takes up the fundamental notions of analytic geometry, with emphasis on the idea of a function, concluding the volume with a full account of limits and continuity. The chapters on analytic geometry include both two and three dimensions, but are confined to problems on lines and planes, leaving all curved lines and surfaces to be treated in the succeeding volumes in connection with the calculus. The chapter on functions is unusually full and thoroughly modern, based on a brief, but clear, exposition of point-sets. In this chapter are included the interpolation formulas of Lagrange and Newton.

The second and third volumes are characterized by the same thoroughness of treatment and soundness of method that I have tried to indicate in this too short account of the first volume. In the second volume, the subject of infinite series receives a full algebraic treatment in a chapter of 75 pages in addition to a long chapter on Taylor's theorem. In the case of functions of several variables, the condition for validity of $\delta^2 f/\delta x \delta y = \delta^2 f/\delta y \delta x$ is adequately presented. The

volume closes with 87 pages on functions of a complex variable, including the fundamental theorem of algebra and conformal representation.

The third volume, on the integral calculus, includes such matters as Cauchy's theorem, the theorem of Gauss on transformation of a volume integral into a surface integral, Green's and Stokes's theorems, with a short discussion of vectors in connection with the latter. There is an admirable chapter on improper integrals, that is, on integrals of discontinuous functions and integrals with infinite limits. The final chapter, on differential equations, is confined to those equations which commonly occur in the applications, but contains a good treatment of the geometric interpretation of a differential equation and the existence proof.

The volumes contain very much more material than could possibly be included in any ordinary university course even in Germany, but at the same time there is scarcely anything which should not be of essential value to every student of physics or of engineering. As a supplementary hand-book, to which the teacher could refer for a sound and clear discussion of fundamental principles, it is all that could be desired, and is quite the best book of this kind, so far at least as the students referred to are concerned, which has come to the reviewer's attention.

M. W. HASKELL.

Functions of a Complex Variable. By E. J. TOWNSEND. New York, Henry Holt and Company, 1915. vii + 384 pp. 8vo. Price $4.00.

THERE has been a noticeable dearth of text-books in the English language on the elements of the theory of functions of a complex variable. The student with an easy command of German and French has found a rich and delightful literature, while his companion still in the period of language difficulties has had little opportunity to choose his reading in function theory in accordance with his individual tastes and requirements. But within the last three years there have appeared English translations of the classic works of Burkhardt and Goursat, a very full volume by Pierpoint, and the book we have before us for review.

The most obvious advantage of Townsend's treatment seems to be the absence of the synoptic character common to

texts written for advanced students of mathematics. Apparently no knowledge of matters beyond the elementary calculus is assumed, unless the existence of the definite integral $\int_a^b f(x)dx$, where $f(x)$ is a continuous function of the real variable x in the interval $a \leqq x \leqq b$, may be regarded as such an assumption. The general theory of the line integral based on the foregoing existence theorem is completely developed. In this connection it may be remarked that the author's use throughout the book of an *ordinary* curve (precedent is cited for the terminology, page 47) as the path of integration makes it practicable to attain complete rigor in the proofs with a minimum of complexity. The ordinary curve has as its most convenient property the quality of being monotone by segments finite in number, a property not possessed by the more general *regular* curve used in Osgood's treatise. To the beginner the fundamental theorems of function theory are fully as satisfactory stated in terms of the simpler curve. The zest for generalization naturally comes at a later period in his study. We have here then a treatment of the function theory which appears to go to the bottom of things without being led into delicate generalizations and which requires on the part of the reader only a thorough first course in calculus and a certain maturity of mind.

The book is unique in its extraordinarily detailed treatment of the elementary functions (55 pages). It thus affords opportunity for an intimate acquaintance with old friends from a new point of view and offers an interesting concrete field of application for many of the general theorems. This is all the more important in view of the fact that "the material chosen deals for the most part with the general properties of functions of a complex variable, and but little is said concerning the properties of some of the more special classes of functions, as for example elliptic functions, etc." It may well be questioned whether a first course in function theory should include among its objects the introduction of a variety of new functions or whether the emphasis should be rather on a more profound acquaintance with the old functions and a clearing up of difficulties naturally arising in the study of the calculus, incidentally laying foundations for work in the field of elliptic functions, etc., if the occasion for such work presents

itself. The latter seems to be distinctly the view-point of the author of the present volume.

Chapter I is devoted to the arithmetic of complex numbers including the geometric representation of the processes of addition, subtraction, multiplication, and division. Chapter II is concerned with the notion of a limit in the complex domain. A number of theorems useful later on are established. For example, it is proved that if $f(z)$ is continuous in a closed region, this continuity is uniform—a theorem not always presented in books of an elementary character. It is in this chapter only that one finds the author yielding to the temptation to generalize beyond the immediate needs of the student. The definition given of continuity in a boundary point of a region is such that the continuity of a function in a closed region does not imply by definition the continuity of the set of boundary values itself. Instead of merely supplying the proof that the set of boundary values is actually continuous, the author states and proves the more general theorem (page 38) that "if $f(z)$ is defined for a closed region S and converges uniformly along an arc C of the boundary of S, then $f(t)$ is continuous, where t denotes the values of z on C." So far as later applications are concerned this theorem seems to be needless. However, the whole chapter is a vigorous exercise in ϵ, δ reasoning and the added refinement may increase its disciplinary value.

Chapter III, a long chapter devoted to the fundamental integral theorems, is one of the most interesting in the book. The existence of a definite integral of a continuous function $f(z)$ over an ordinary curve C having been once established by means of the relation $\int_C f(z)dz = \int_C udx - vdy + i\int_C vdx + udy$, $f(z)$ is thereafter kept intact. The Cauchy-Riemann differential equations are not given until near the end of the chapter and are then derived by means of the very integral theorems which usually depend on them. The development of the integral theorems starts with the Goursat proof of the Cauchy theorem, or as the author calls it, the Cauchy-Goursat theorem, a title which seems proper since Goursat really contributed a new theorem rather than a new proof of an old theorem. Then follows the derivation of the Cauchy integral formula and the proof that the mere existence of a derivative of a

function of a complex variable in a region implies the continuity of the derivative.

Morera's converse of the Cauchy theorem is given particular prominence and finds wide application. Using this theorem it is easily shown that the necessary and sufficient condition for a continuous function $f(z) = u + iv$ to be holomorphic is either that $f(z)$ satisfy the integral equation $\int_C f(z)dz = 0$ or that u and v have continuous first partial derivatives satisfying the Cauchy-Riemann differential equations. The integral theorem renders unnecessary the usual ϵ, δ proof of the sufficiency of the differential equations. The chapter closes with the usual theorems concerning Laplace's differential equation and a brief discussion of its significance in mathematical physics.

In Chapter IV the elementary functions are mapped in greater detail than in any other English text with which the reviewer is familiar. The definition of the exponential function as

$$e^z = \lim_{n=\infty} \left(1 + \frac{z}{n}\right)^n$$

and the definitions of the sine and cosine functions as

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2},$$

permit the discussion of the properties of these functions in advance of the introduction of power series. In addition to the usual elementary functions several particular cases of the function

$$w = \log \frac{(z - \alpha_1)^{k_1}(z - \alpha_2)^{k_2} \cdots (z - \alpha_n)^{k_n}}{(z - \beta_1)^{\lambda_1}(z - \beta_2)^{\lambda_2} \cdots (z - \beta_n)^{\lambda_n}}$$

having important physical applications are mapped and the physical interpretations pointed out.

Chapter V, on linear fractional transformations, is sufficiently full to serve as a foundation for the theory of automorphic functions. The figures in illustration of the process of stereographic projection are unusually clear and accurate and should be a great aid at this point to the student's imagination.

Chapter VI, on infinite series, can be criticized only on the ground that it comes so late (approximately at the beginning

of the second half of the volume). The author's purpose seems to have been to do first everything that can be conveniently done without the use of power series and then, having once introduced power series, to make rapid and continual use of them in developing the general analytic function theory. Logically this method is beyond criticism and it may have pedagogical advantages. In this chapter the usual theorems on power series are developed, closing with Taylor's expansion of a function holomorphic in a given region.

Chapter VII takes up the theorem on Taylor's expansion and out of it develops the notion of analytic continuation and the general definition of an analytic function. This discussion is as clear cut and accurate as one often finds. Particularly to be noted, as serving to dispel any hazy notions the student may have at this stage, is the comparison by means of examples of the properties of functions of a real variable which are differentiable infinitely many times with the properties of analytic functions of a complex variable—also differentiable infinitely many times. For example, the function

$$f(x) = e^{-1/x^2},$$

considered zero when x is zero, has infinitely many derivatives in the real domain for all real values of x. Nevertheless

$$\lim_{x=0} \frac{f(x)}{x^k} = 0,$$

where k is any integer however large. This it is pointed out is in sharp contrast to the behavior of a single valued analytic function at a zero point, in that the zero points of the latter are necessarily of definite positive integral order. Several examples, such as the series

$$E(z) = \frac{z}{1-z^2} + \frac{z^2}{1-z^4} + \frac{z^4}{1-z^8} + \cdots,$$

convergent except for values of z upon the unit circle, are given to make clear the distinction between an analytic function and a function defined by an analytic expression.

The notion of analytic function having been established, zero points, poles, and essential singular points are defined and the usual theorems are included. Regular and singular points at infinity, Laurent's expansion, and the theory of

residues are next in order. We then have a full discussion of the properties of rational functions, with a proof of the fundamental theorem of algebra. Passing to transcendental functions, the Mittag-Leffler theorem in its simplest form is proved. An innovation in connection with this proof consists in the introduction of a figure as an aid to the reader in following the analytic reasoning. The function $\zeta(z)$ and the elliptic function $\wp(z) = -\zeta'(z)$ are used in illustration of the theorem. Such properties of infinite products as are essential to an understanding of Weierstrass's primary factors are developed. The chapter closes with a brief discussion of the properties of simply and doubly periodic functions.

Chapter VIII, the last, is devoted to a brief but fairly comprehensive treatment of the properties of multiple-valued functions.

On the whole the book is well coordinated with our undergraduate courses and covers just about the ground in function theory which the first year graduate student of mathematics should get well in hand.

H. B. PHILLIPS.

Plane and Spherical Trigonometry and Tables. By G. WENTWORTH and D. E. SMITH. Boston, Ginn and Company, 1914. 230 + 104 pp.

QUOTING from the preface, this is "a work to replace the Wentworth Trigonometry which has dominated the teaching of the subject in America for a whole generation." . . . "With respect to sequence the rule has been followed that the practical use of every new feature should be clearly set forth before the abstract theory is developed."

In several particulars the book could be made more useful for students intending to pursue mathematics further. For instance, no mention has been made of Argand's diagram or of hyperbolic functions, though the logarithms of negative numbers are unusually well treated. This excellence is balanced by the unfortunate use of negative characteristics which will lead the student into endless trouble later.

Inverse function theory merits more extensive treatment even at the cost of fewer examples, but the related general formulas for all angles having the same sine, cosine or tangent are to be commended. The small pink representations of coordinate paper are attractive to the eye, but the graphs of

the trigonometric functions are too small and not accurate. One full page graph with carefully distinguished curves would be more satisfactory. In the early pages the definition of coordinates is correct, but unfortunately all notion of directed lines is abandoned in the proofs of the summation formulas for sines and cosines.

Reduction of functions to the first quadrant, which in many books is needlessly diffuse, is here given in a compact form which lacks only a good rule as a summary to be quite sufficient. The double inequality in the proof for the sines of small angles follows the usual course, and as in most of our text-books does not show that the two tangents are longer than the arc. The corresponding S and T tables have been made relatively inconspicuous and supplemented by more useful tables of functions of small angles given for each second.

In the discussion of variation of trigonometric functions double signs have been used to advantage with the zero values as well as with the infinity sign. The solid perspective figures in the latter part of the book are good, and at the end is found, what is none too common in our text-books, an index.

As a whole the book does not differ noticeably from the many text-books covering the same ground. Examples are particularly abundant while the theoretical portions might well be amplified and improved.

F. H. SAFFORD.

Wilhelm Schells Allgemeine Theorie der Kurven Doppelter Krümmung. Dritte Auflage neu bearbeitet von ERICH SALKOWSKI. Leipzig, B. G. Teubner, 1914. 196 pp. Price 8 marks.

THE first edition of this book appeared in 1859, the second in 1898. It was the purpose of the author to develop the elements of the theory of curves of double curvature by the geometric method of infinitesimals without recourse to higher analysis. Since the investigation included the study of the three surfaces generated by the tangents, principal normals, and binormals of a general curve, there appears incidentally a considerable amount of material concerning the geometry of a surface in the neighborhood of an ordinary point. Consequently a reader (who might easily be an undergraduate) could obtain from the book a very good idea of the elements of the differential geometry of curves and surfaces.

Any one familiar with Salkowski's papers in this field would recognize his fitness to prepare a third edition of the book. For a number of his papers are written from its point of view. Many sections have been rewritten and there are a number of essential additions. A notable one is the proof of the fundamental theorem concerning the determination of a curve by its intrinsic equations. A new chapter is devoted to a full discussion of skew ruled surfaces, in preparation for the study of those generated by the principal normals and binormals of a curve.

When two curves in space are in one-to-one correspondence such that corresponding tangents, principal normals, and binormals respectively are parallel, the curves are in the relation of a transformation of Combescure, to use the terminology due to Bianchi. One readily thinks of other correspondences involving parallel arrangements, as well as ones having to do with differential quantities. An investigation of these questions appears in the new edition for the first time, as do also the ideas and properties of successive evolutes.

A student reading one of the advanced treatises on differential geometry will do well to consult this little book, because of its method of attack and its richness in material.

LUTHER PFAHLER EISENHART.

Ten British Mathematicians. By ALEXANDER MACFARLANE. No. 17, Mathematical Monographs edited by Mansfield Merriman and Robert S. Woodward. New York, John Wiley and Sons, 1916. 148 pp. Price $1.25.

THIS is somewhat of a departure from the preceding sixteen numbers of this series in subject matter, being an account of individual mathematicians and their contributions to mathematics, rather than a survey of some particular field of the subject. The ten men are George Peacock, Augustus De Morgan, Sir William Rowan Hamilton, George Boole, Arthur Cayley, William Kingdon Clifford, Henry John Stephen Smith, James Joseph Sylvester, Thomas Penyngton Kirkman, and Isaac Todhunter. The author had the advantage of personal acquaintance with a number of the men of whom he wrote, as well as the interest of having been born a fellow countryman. He has given us the life history of the man without too much detail, and yet with enough intimacy, so that we have a picture of the man as a man, as well as a scientist.

The book consists of ten lectures delivered to audiences composed of students, instructors, and townspeople at Lehigh University during the years 1901 to 1904. As each was delivered as a separate lecture, there is no direct connection between them. The critical analysis of the contributions of each mathematician is probably as well done as would be possible considering the character of the audience and the length of time at the disposal of the speaker. We get a very good general notion of what each contributed and its value to the science.

The ten men chosen are not equally well known to American mathematicians. Certainly we have a deeper interest in and knowledge of Sylvester than of the others. Todhunter would be best known for his many textbooks, and De Morgan for his Paradoxes. It is doubtful if American mathematicians know the English as well as the continental mathematicians, especially the German. American students of mathematics who have gone abroad for study have gone chiefly to the German universities. This has tended to bring American mathematicians into closer relations with the German mathematicians and their work than with the English. For that reason, this book on English mathematicians is valuable.

There is a tendency today to try to give some comprehension of the problems and methods and achievements of mathematics to those who are not specialists in mathematics. A recent book by Professor G. A. Miller, of the University of Illinois, is such an effort. These lectures by Dr. Macfarlane serve that purpose, as they were not prepared for specialists in mathematics alone.

We have a partial promise in the preface that there will be a later volume dealing with ten mathematicians who worked chiefly in applied mathematics. We hope that this promise may be speedily fulfilled.

The reading of the page proofs does not seem to have been done carefully enough. There are too many letters that have dropped out. Otherwise this volume has the same convenient size and neatness of appearance that characterize the other volumes of this set.

THOS. E. MASON.

Das Perpetuum Mobile. By F. ICHAK. Bändchen 462 of the series: Aus Natur und Geisteswelt. Leipzig, Teubner, 1914. 104 pp. Price M. 1.25

THE question of perpetual motion has interested mankind for many centuries. The author says: "We see before us an almost endless series of attempts of different sorts which have one thing in common—a negative result. But mankind does not let itself be convinced of the impossibility of the undertaking by these unnumbered failures: here the wish is mightier than the judgment."

This little book does not claim to be an exhaustive treatise on the subject, but does cover the field remarkably well in such short space. This is largely due to the systematic arrangement of the material. Perpetual motion is discussed under three heads: first, that perpetual motion which depends upon the creation of energy by some device, and this is what is usually meant by perpetual motion; second, that which does not create energy but transforms it from one form to another in some cycle; and third, that motion which is not perpetual but of indefinite length, like the motion of the radium clock. The first of these is the oldest and has been the goal of the greatest number of perpetual motion seekers. The efforts to make a device for the creation of energy which could be utilized in doing work are discussed under several types, such as mechanical, magnetic, hydraulic perpetual motion, etc. Type machines and machines which have attracted unusual attention are described in detail and often illustrated. The effect of learning and the development of science upon the efforts at finding such a machine are discussed as well as the reaction of these experiments on science. With the theory of the conservation of energy came the attempts to secure perpetual motion of the second sort mentioned. The perpetual motion machines of the third sort, or apparent perpetual motion machines, like the radium clock or the radiometer, do not create energy but have it furnished from nature, and while the supply is very great it is not inexhaustible. The author closes with a section on perpetual motion in the future, in which among other things is mentioned the possibility of getting around the difficulties of the conservation of energy by a new theory concerning energy. There is also a prophecy, likely to be fulfilled, that we shall always have the perpetual motion seekers with us.

THOS. E. MASON.

Constructive Geometry. Exercises in elementary geometric drawing. Prepared under the direction of EARLE RAYMOND HEDRICK. New York, Macmillan Company, 1916. 4to. 72 pages; 43 figures.

THIS book is a manual in very elementary work in geometric drawing. The scheme is first a discussion of a problem, then a number of exercises, finally a number of blank pages on which to draw the exercises. It begins with divisions of a straight line, and emphasizes neatness and accuracy in each detail.

Exercises on the circle include a considerable number in design, all of which are very elementary. While this drill is in the right direction, it is almost too simple to be of much value. In fact one of the greatest needs of our instruction in geometry is a more comprehensive treatment of construction by means of ruler and compass, including a full discussion of the graphical solution of quadratic equations.

VIRGIL SNYDER.

Grundzüge der Perspektive nebst Anwendungen. By Professor KARL DOEHLEMANN, of the technical school at Munich. (Aus Natur und Geisteswelt, volume 510.) Leipzig, Teubner, 1916. 104 pages, 91 figures, and 11 photographs.

THE volumes of this series are to be written in a popular style, suitable for readers with little or no previous training in the specific line discussed by each volume. Thus, in the present case no use is made of solid geometry, and only the most elementary properties of similar polygons in plane geometry are assumed as known. Probably for this reason the introductory chapter strikes a reader as heavy and prolix, although the method of introduction is unusually well chosen. The author finds it necessary to present the essentials of the $h - v$ orthogonal projection, but only in so far as points and straight lines are concerned, and these are employed immediately to produce a perspective picture.

The concept of the vanishing point and vanishing line is developed in great detail. No exercises are proposed for the reader, except that an occasional construction is left to be completed, but on the other hand a large number of drawings and several reproductions of paintings are given, to assist in illustrating the point under discussion. Apart from the very brief explanation of the two-plane orthogonal projection, the

discussion of figures bounded by planes is treated in sufficient detail. A short discussion of the perspective of the circle is added, and a still shorter discussion of shades and shadows. The book has a table of contents, an index, a list of books for further reading, and is almost entirely free from typographical errors. It is well fitted for its purpose of providing a knowledge of the essentials of perspective in a small popular volume.

VIRGIL SNYDER.

A Treatise on Statics. By GEORGE M. MINCHIN. Vol. II. Fifth Edition. Revised by H. T. GERRANS. Oxford, The Clarendon Press, 1915.

"IN December, 1913, when the preparations for a new edition became necessary, it was suggested to the author (1) that certain portions should be omitted (in the hope that they might form the basis of a separate work); (2) that some account of the author's recent researches in spherical harmonics should be given; (3) that a substantial number of examples should be added." The author determined which chapters and articles should be left out, but his death occurred before the task of revision was begun.

In this latest edition 467 examples have been added in an appendix, also a very few in the text, while the original numbering of the articles in the fourth edition has been retained. Three entire chapters (of the fourth edition) have been omitted, Chapter XVIII, Analysis of Strains and Stresses, Chapter XIX, Electrostatics, also the chapter on Astatic Equilibrium, viz., Chapter XIV, in which the subject was treated with the aid of quaternions.

Other omissions are arts. 221–226, 228–234, on reciprocal screws, etc., arts. 280, 281, on general properties of static energy and stable configuration, arts. 288–296 on elastic wires, inextensible surfaces and liquid surfaces, arts. 307–314 on tortuous curves and kinetic analogues. Note A, on the equation of capillarity, has disappeared, as well as the index to Volume II.

Otherwise this edition is a verbatim reproduction of the fourth edition, with occasional obvious omissions of subscripts. In art. 235 the correct reading is $\phi = qAR$ and $K = pR$. In the last equation in art. 241, the $-$ sign should be $+$, and in equation (2) of problem 9 at the end of the same article α/μ should be a/μ.

Certain errata found in the fourth edition, all but one of which were noted by Professor Hoover of Ohio University, have been corrected, except that a superfluous 1 still remains in the answer to problem 12, art. 325. Other errors uncorrected in both editions are $W/2$ in place of W as the answer to problem 5, art. 205; and in art. 227 the variables should be connected by *three* equations if the body has three degrees of freedom.

It is evident that this edition is essentially a reprint, and that the reviser felt a commendable reluctance in changing in any way, except as sanctioned by the author, a text which holds such a high reputation among scholars.

F. H. SAFFORD.

NOTES.

AT the meeting of the Edinburgh mathematical society on November 10 the following papers were read: By E. T. WHITTAKER: "Some theorems on determinants"; by H. DATTA: "On the theory of continued fractions."

THE National academy of sciences held its autumn meeting at the Massachusetts Institute of Technology on November 13–15. The programme included the following mathematical papers: By F. R. MOULTON: "On analytic functions of infinitely many variables"; by H. S. WHITE, F. N. COLE, and LOUISE D. CUMMINGS: "Enumeration of all triad systems on fifteen elements"; by W. E. STORY: "Some variable three-term scales of relation." The list of scientific exhibits included graphic representations of triad systems, by H. S. WHITE.

THE Association of mathematics teachers of New Jersey held its fifth regular meeting at Newark on November 25th. The programme included the final report of the committee on trigonometry courses and papers by: A. W. BELCHER: "The first year high school course in mathematics"; W. D. REES: "Newton's analytical triangle"; C. R. MACINNES: "Some theorems on regular polygons described on the sides of a triangle"; WILLIAM STRADER: "Teaching first year algebra."

THE two Benjamin Peirce instructorships in mathematics at Harvard University (see BULLETIN, volume 21, page 315) are again open to general competition. Applications for the year 1917–1918, accompanied by the necessary papers, should be sent to Professor Bôcher, who will be glad to furnish further particulars. All applications must be filed by February 1, 1917.

A complete list of American doctorates conferred during the academic year 1915–1916 has been compiled for *School and Society*. From advanced proof sheets kindly furnished by Professor Cattell it appears that the total number during this period was 603, of which 327 may be assigned to the sciences. The list of mathematical doctorates, 34 in number, follows below, the title of the dissertation being given in each case.

J. W. ALEXANDER, II, Princeton, "Functions which map the interior of the unit circle upon simple regions"; P. M. BATCHELDER, Harvard, "The hypergeometric difference equation"; R. W. BRINK, Harvard, "Some integral tests for the convergence and divergence of infinite series"; A. F. CARPENTER, Chicago, "Ruled surfaces whose flecnode curves have plane branches"; L. C. COX, Cornell, "The finite groups of birational transformations of a net of cubics"; J. V. DEPORTE, Cornell, "Irrational involutions on algebraic curves"; W. VAN N. GARRETSON, Michigan, "On the asymptotic solution of the non-homogeneous linear differential equation of the nth order. A particular solution"; R. E. GILMAN, Princeton, "On the summability by the Cesaro method of partial derivatives of Laplace series"; A. M. HARDING, Chicago, "On certain loci projectively connected with a given plane curve"; W. L. HART, Chicago, "Differential equations and implicit functions in infinitely many variables"; M. G. HASEMAN, Bryn Mawr, "Knots with a census of the amphicheiral with twelve crossings"; J. O. HASSLER, Chicago, "Plane nets periodic of period three under the Laplace transformation"; ARCHIBALD HENDERSON, Chicago, "The twenty-seven lines upon the cubic surface"; J. R. KLINE, Pennsylvania, "Double elliptic geometry in terms of point and order"; PEYSAH LEYZERAH, Clark, "On the indeterminate linear inequality with irrational coefficients"; G. H. LIGHT, Yale, "The dependence of the topography of envelopes of systems of ex-

tremals on curvature"; P. H. LINEHAN, Columbia, "Contributions to equilong geometry"; F. J. MCMACKIN, Columbia, "Some theorems in the theory of summable divergent series"; A. L. MILLER, Harvard, "Systems of pencils of lines in ordinary space"; NORMAN MILLER, Harvard, "Some problems connected with the linear connectivity of manifolds"; F. D. MURNAGHAN, Johns Hopkins, "The lines of electric force due to a moving electron"; J. R. MUSSELMAN, Johns Hopkins, "A set of eight self-associated points in space"; A. L. NELSON, Chicago, "Plane nets with equal Laplace-Darboux invariants"; T. A. PIERCE, California, "The numerical factors of the arithmetic forms"; C. H. RAWLINS, JR., Johns Hopkins, "Complete systems of concomitants of the three-point and the four-point in elementary geometry"; S. W. REAVES, Chicago, "Metric properties of flecnodes on ruled surfaces"; A. R. SCHWEITZER, Chicago, "Les idées directrices de la logique génétique des mathématiques"; T. M. SIMPSON, Wisconsin, "On a functional equation of Abel"; E. S. SMITH, Virginia, "Solution of the equation of secular variation by a method due to Hermite"; PAULINE SPERRY, Chicago, "Properties of a certain projectively defined two-parameter family of curves on a general surface"; J. H. WEAVER, Pennsylvania, "Some extensions of the work of Pappus and Steiner on tangent circles"; MARY E. WELLS, Chicago, "The determination of all inequalities of certain types in general linear integral equation theory"; A. R. WILLIAMS, California, "A birational transformation connected with a pencil of cubics"; C. H. YEATON, Chicago, "Surfaces characterized by certain properties of their directrix congruences."

THE Royal Society has awarded the Sylvester medal to Professor GASTON DARBOUX, of the University of Paris.

DR. A. SIGNORINI, of the University of Padua, has been appointed associate professor of rational mechanics at the University of Palermo.

AT the U. S. Naval Academy DR. R. E. ROOT has been promoted to a full professorship of mathematics and mechanics.

PROFESSOR J. F. REILLY, of the State University of Iowa, has been promoted to an associate professorship of mathematics.

AT the University of Washington Dr. E. T. BELL has been promoted to an assistant professorship of mathematics.

AT Indiana University Dr. CORA B. HENNEL has been promoted to an assistant professorship of mathematics.

MR. C. W. WESTER, of the State University of Iowa, has been appointed assistant professor of mathematics in the Iowa State Teachers College.

DR. D. F. BARROW, of the University of Georgia, has been appointed instructor in mathematics in the Sheffield Scientific School.

PROFESSOR R. M. BARTON, of Lombard College, has been appointed instructor in mathematics in the University of Minnesota.

DR. A. L. MILLER has been appointed instructor in mathematics in the University of Michigan.

PROFESSOR G. PENNACCHIETTI, of the University of Catania, died August 21 at the age of sixty-six years.

ASSISTANT professor J. C. RAYWORTH, of Washington University, died November 11 after an illness of several months.

DR. HENRY GUNDER, formerly professor of mathematics at Findlay College, died November 25 at the age of seventy-nine years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

HOPKINS (L. A.). See ZIWET (A.).

LECAT (M.). Table des matières des tomes I à XX (1894–1913) de *L'Intermédiaire des Mathématiciens*. Paris, Gauthier-Villars, 1916. 8vo. 4+280 pp. Fr. 8.00. To subscribers to *L'Intermédiaire* for 1915 and 1916, Fr. 4.00.

LOREY (W.). Das Studium der Mathematik an den deutschen Universitäten seit Anfang des 19. Jahrhunderts. (Abhandlungen über den mathematischen Unterricht in Deutschland veranlasst durch die internationale mathematische Unterrichtskommission, Band III, Heft 9.) Leipzig, Teubner, 1916. Gr. 8vo. 16+440 pp. Geh. M. 12.00

LOVE (C. E.). Differential and integral calculus. New York, Macmillan, 1916. 8vo. 18+343 pp. Cloth. $2.10

RIGGS (N. C.). Analytic geometry. 9th reprint. New York, Macmillan, 1915. 12mo. 294 pp. $1.60

ZIWET (A.) and HOPKINS (L. A.). Analytic geometry and principles of algebra. 3d reprint. New York, Macmillan, 1915. 12mo. 369 pp. $1.60

II. ELEMENTARY MATHEMATICS.

ANDOYER (H.). Nouvelles tables trigonométriques fondamentales contenant les valeurs naturelles des lignes trigonométriques de centième en centième du quadrant avec vingt décimales, de neuf en neuf minutes avec dix-sept décimales, et de dix en dix secondes avec quinze décimales. Ouvrage publié à l'aide d'une subvention accordée par l'Université de Paris (Fondation Commercy). 2 tomes. Paris, Hermann, 1915–1916. 4to. Tome 1. 18+342 pp. Tome 2. 3+275 pp. Fr. 24.00+18.00

BALTIN (R.). See MÜLLER (H.).

BARDEY (E.). See LIETZMANN (W.).

BRUFF, HAYDEN and WATKINS. Practical primary arithmetic. Philadelphia, Lippincott, 1916.

ELSON (W. H.). See SHORT (R. L.).

HAYDEN. See BRUFF.

LIETZMANN (W.) und BARDEY (E.). Aufgabensammlung für Arithmetik, Algebra und Analysis. Reformausgabe für höhere Knabenschulen von W. Lietzmann und P. Zühlke. A: für Gymnasien. 1ter Teil: Unterstufe, von W. Lietzmann. 2te Auflage. Leipzig, Teubner, 1915. Gr. 8 vo. 6+202 pp. Geb. M. 2.00

MAHLERT (A.). See MÜLLER (H.).

MAIWALD (W.). See MÜLLER (H.).

MÜLLER (H.). Die Mathematik auf den Gymnasien und Realschulen. Ausgabe A: für Gymnasien und Progymnasien. 1ter Teil: Die Unterstufe. 6te Auflage. Leipzig, Teubner, 1915. Gr. 8vo. 8+144 pp. Geb. M. 2.20

——. Die Mathematik auf den Gymnasien und Realschulen. Ausgabe B: für reale Anstalten und Reformschulen. 2ter Teil: Die Oberstufe. 1te Abteilung: Planimetrie, Algebra, Trigonometrie und Stereometrie. 5te Auflage. Leipzig, Teubner, 1915. Gr. 8vo. 11+320 pp. Geb. M. 3.00

MÜLLER (H.), BALTIN (R.) und MAIWALD (W.). Kurzgefasstes Lehrbuch der Mathematik für Seminare und Präparandenanstalten. 5te Auflage. Leipzig, Teubner, 1915. Gr. 8vo. 10+256 pp. Geb. M. 2.60

MÜLLER (H.), BALTIN (R.) und SEGGER (F.). Rechenbuch für Präparandenanstalten. 1ter Teil· Lehraufgabe der 3te Klasse. 4te Auflage. Leipzig, Teubner, 1915. Gr. 8vo. 6+184 pp. Geb. M. 1.80

MÜLLER (H.) und MAHLERT (A.). Mathematisches Lehr- und Uebungsbuch für Lyzeen (höhere Mädchenschulen). 1ter Teil: Arithmetik und Algebra. 5te Auflage. Leipzig, Teubner, 1915. Gr. 8vo. 4+160 pp. Geb. M. 2.00

MÜLLER (H.) und PIETZKER (F.). Rechenbuch für die unteren Klassen der höheren Lehranstalten. Ausgabe in 3 Heften. Heft 3: für Quarta. 4te Auflage. Leipzig, Teubner, 1915. Gr. 8vo. 6+122 pp. M. 1.00

MÜLLER (H.) und SEGGER (F.). Rechenbuch für die Vorschule. Heft 1: Lehraufgabe des ersten Schuljahres. 5te Auflage. Leipzig, Teubner, 1915. 72 pp. Gr. 8vo. M. 0.80

PHILIP (M.). Algebra. New York, Hinds, Hayden and Eldredge, 1916. 8vo. 10+201 pp.

PIETZKER (F.). See MÜLLER (H.).

SAMPSON (C. H.). Algebra review. New York, World Book Company, 1916. 8vo.

SEGGER (F.). See MÜLLER (H.).

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